

MAS317 Linear Algebra II, 2007
Solutions to Coursework 7.

1. (i) $f((3, 1+i), (4i, 2-i)) = (3, 1+i) \begin{pmatrix} 2 & 1+i \\ 3 & 0 \end{pmatrix} \begin{pmatrix} -4i \\ 2+i \end{pmatrix}$
 $= (9+3i, 3+3i) \begin{pmatrix} -4i \\ 2+i \end{pmatrix} = 15 - 27i.$ [5 marks]

(ii) $(f : \mathcal{B}) = (M_{\mathcal{B}}^{\mathcal{E}})^T(f; \mathcal{E})(\overline{M_{\mathcal{B}}^{\mathcal{E}}}) = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 2 & 1+i \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$
 $= \begin{pmatrix} 2+3i & 1+i \\ 3+2i & -1+i \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 3+2i & 4-i \\ 4+3i & 1-2i \end{pmatrix}.$ [10 marks]

2. With notation in (6.7), let $\mathcal{B} = v_1, \dots, v_n$ and let $\mathcal{C} = w_1, \dots, w_n$.

Let $U = \langle v_1, \dots, v_p \rangle$ and let $W = \langle w_{p'+1}, \dots, w_n \rangle$.

Then $n \geq \dim(U + W) = p + n - p' - \dim(U \cap W)$, and so if $\dim(U \cap W) = 0$, then $p \leq p'$.

[5 marks]

Now consider $u \in (U \cap W)$.

As $u \in U$, $u = a_1 v_1 + \dots + a_p v_p$ for some $a_1, \dots, a_p \in \mathbb{C}$ and so

$$f(u, u) = (a_1, \dots, a_p, 0, \dots, 0)(f; \mathcal{B})(\overline{a_1}, \dots, \overline{a_p}, 0, \dots, 0)^T$$

$$= a_1 \overline{a_1} d_1 + \dots + a_p \overline{a_p} d_p.$$

Now $d_i > 0$ for $1 \leq i \leq p$, and so $a_i \overline{a_i} d_i = |a_i|^2 d_i$ and is real and positive unless $a_i = 0$, when it is 0.

[5 marks]

Hence $f(u, u) \geq 0$ and is equal to 0 only if $u = \underline{0}$. (*)

As $u \in W$, $u = b_{p'+1} w_{p'+1} + \dots + b_n w_n$ for some $b_{p'+1}, \dots, b_n \in \mathbb{C}$ and so

$$f(u, u) = (0, \dots, 0, b_{p'+1}, \dots, b_n)(f; \mathcal{C})(0, \dots, 0, \overline{b_{p'+1}}, \dots, \overline{b_n})^T$$

$$= b_{p'+1} \overline{b_{p'+1}} e_{p'+1} + \dots + b_n \overline{b_n} e_n.$$

Now $e_i \leq 0$ for $p' \leq i \leq n$ and so $b_i \overline{b_i} e_i = |b_i|^2 e_i$ and is real and ≤ 0 .

[5 marks]

Hence from (*) $f(u, u) = 0$ and so $u = \underline{0}$. So $U \cap W = \{\underline{0}\}$ and so $p \leq p'$.

Similarly $p' \leq p$ and so $p = p'$. Now $p + q = p' + q'$ as these are the row ranks of the two matrices and they are congruent and so have the same row rank. Hence also $q = q'$.

[5 marks]

3(a). A is Hermitian congruent to B if there exists an invertible matrix P such that $B = \overline{P^T}AP$. [5 marks]

$$A = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ for some } a, b, c, d \in \mathbb{C}.$$

$$\text{So } A = \begin{pmatrix} -\bar{a} & 0 \\ -\bar{b} & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a\bar{a} & -b\bar{a} \\ -a\bar{b} & -b\bar{b} \end{pmatrix}. \quad [10 \text{ marks}]$$

Now the diagonal entries are $-a\bar{a} = -|a|^2$ and $-b\bar{b} = -|b|^2$ and so are real and ≤ 0 . [5 marks]

(b). A matrix Hermitian congruent to a given matrix results when an elementary row operation e and the complex conjugate $\overline{e^T}$ of the corresponding column operation are applied. [3 marks]

(i) $R_1 \rightarrow iR_1, C_1 \rightarrow iC_1$ gives A congruent and not Hermitian congruent to B . [3 marks]

(ii) $R_1 \rightarrow R_1 + (2+i)R_2, C_1 \rightarrow C_1 + (2-i)C_2$ gives A Hermitian congruent and not congruent to B . [3 marks]

(iii) $R_1 \rightarrow R_1 + 2R_2, C_1 \rightarrow C_1 - 2C_2$ gives A not congruent and not Hermitian congruent to B . [3 marks]

(iv) $R_1 \rightarrow R_1 + 2R_2, C_1 \rightarrow C_1 + 2C_2$ gives A congruent and Hermitian congruent to B . [3 marks]

[For (iii) and (iv) NOTE THAT the complex conjugate of 2 is 2 and not -2.]

4. A matrix A is *Hermitian* if $A = \overline{A}^T$. [2 marks]

(i) In an Hermitian matrix A , $A_{ii} = \overline{A_{ii}}$ and so A_{ii} is real. [3 marks]

(ii) If A is Hermitian, then $A = \overline{A}^T$, and if B is Hermitian congruent to A then A is Hermitian congruent to B by (6.9) and so $B = \overline{P}^T A P$ for some invertible P . So $\overline{B}^T = (\overline{\overline{P}^T A P})^T = \overline{P}^T A^T (\overline{P}^T)^T = \overline{P}^T A^T P = \overline{P}^T A P = B$ so B is Hermitian. [5 marks]

5. $A \begin{pmatrix} 1 & 2i \\ -2i & 4 \end{pmatrix} R_2 \rightarrow R_2 + 2iR_1 \begin{pmatrix} 1 & 2i \\ 0 & 0 \end{pmatrix} C_2 \rightarrow C_2 - 2iC_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
 A has rank 1, signature 1. [3 marks]

$B \begin{pmatrix} 0 & -3i \\ 3i & 0 \end{pmatrix} R_1 \rightarrow R_1 + iR_2 \begin{pmatrix} -3 & -3i \\ 3i & 0 \end{pmatrix} C_1 \rightarrow C_1 - iC_2 \begin{pmatrix} -6 & -3i \\ 3i & 0 \end{pmatrix}$
 $R_2 \rightarrow R_2 + (i/2)R_1 \begin{pmatrix} -6 & -3i \\ 0 & 3/2 \end{pmatrix} C_2 \rightarrow C_2 - (i/2)C_1 \begin{pmatrix} -6 & 0 \\ 0 & 3/2 \end{pmatrix}$
 B has rank 2, signature 0. [6 marks]

$C \begin{pmatrix} -1 & 1+i \\ 1-i & -2 \end{pmatrix} R_2 \rightarrow R_2 + (1-i)R_1 \begin{pmatrix} -1 & 1+i \\ 0 & 0 \end{pmatrix}$
 $C_2 \rightarrow C_2 + (1+i)C_1 \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$
 C has rank 1, signature -1. [3 marks]

$D \begin{pmatrix} 0 & 2+i \\ 2-i & 0 \end{pmatrix} R_1 \rightarrow R_1 + R_2 \begin{pmatrix} 2-i & 2+i \\ 2-i & 0 \end{pmatrix} C_1 \rightarrow C_1 + C_2 \begin{pmatrix} 4 & 2+i \\ 2-i & 0 \end{pmatrix}$
 $R_2 \rightarrow R_2 - ((2-i)/4)R_1 \begin{pmatrix} 4 & 2+i \\ 0 & -5/4 \end{pmatrix} C_2 \rightarrow C_2 - ((2+i)/4)C_1 \begin{pmatrix} 4 & 0 \\ 0 & -5/4 \end{pmatrix}$
 D has rank 2, signature 0. [8 marks]